Extremal graph theory, Stability, and Anti-Ramsey theorems

Miklós Simonovits

Alfréd Rényi Mathematical Institute of the Hungarian Academy of Sciences, Budapest, Hungary

Extremal graph theory is one of the most developed branches of Discrete Mathematics. Stability methods introduced primarily by the author [5] are very successful to prove sharps results in this field. We shall give some illustration of this method for graphs, hypergraphs, (among others, the Füredi-Simonovits and Füredi-Pikhurko-Simonovits theorems obtained by the Stability method. We shall also apply stability methods to Anti- Ramsey problems. An Anti-Ramsey problem is where a sample graph L is fixed and we colour the edges of , e.g., a complete graph K_n without having a copy of L in which all the edges have distinct colours. Several problems in combinatorics can be reduced to extremal graph problems. Erdős, Simonovits and Sós [4] basically reduced certain Anti- Ramsey problems to extremal graph problems. In the last part of the lecture we shall consider Dual Anti-Ramsey problems, coming from Theoretical Computer Science. Burr, Erdős, Graham and T. Sós [1] defined and investigated a dual variant of the Anti-Ramsey problems. They wrote up some of their results also in a second paper joint with Peter Frankl [2]. As they pointed out, one of the most interesting cases they could not settle was that of C_5.

The dual Anti-Ramsey problem. Let us fix a sample graph L, and consider a (variable) graph G_n on n vertices, with

$$e = e(Gn) > \mathbf{ex}(n, L)$$

edges. Let $\chi_S(G_n, L)$ denote the minimum number of colours needed to colour the edges of G_n so that no $L \subset G_n$ has two edges of the same colour. Determine

$$\chi_S(n,e,L) := \min\{\chi_S(Gn,L) : e(Gn) = e\}.$$

Here we improve several results of [1] and [2]. We shall prove, among others, that if a graph G_n has $e = \left\lfloor \frac{1}{4}n^2 \right\rfloor + 1$ edges and we colour its edges so that every $C_5 \subset G_n$ is 5coloured, then we have to use at least $\left\lfloor \frac{n}{2} \right\rfloor + 3$ colours, if n is sufficiently large. This result is sharp.

Theorem 1 *There exists a threshold n_0 such that if $n > n_0$, and a graph G_n has $\left\lfloor \frac{1}{4}n^2 \right\rfloor + 1$ edges and we colour its edges so that every C_5 is 5coloured, then we have to use at least $\left\lfloor \frac{n}{2} \right\rfloor + 3$ colours.*

Theorem 2 *There exists a function $\theta(n) \to \infty$ such that if $0 < k \le \binom{h}{2} < \theta(n)$, then the upper bound of Theorem 4.2/[1] is sharp for $e = \left\lfloor \frac{1}{4}n^2 \right\rfloor + k$:*

$$\chi_S(n,e,C_5) = (h+1)\left\lfloor \frac{n}{2} \right\rfloor + k.$$

Because of the monotonicity, this implies

Theorem 3 *There exists a function $\theta(n) \to \infty$ such that if $0 < k \le \binom{h}{2} < \theta(n)$, then for $e = \left\lfloor \frac{1}{4}n^2 \right\rfloor + k$*

$$\chi_S(n,e,C_5) = (h+1)\left\lfloor \frac{n}{2} \right\rfloor + k + O(\sqrt{k}).$$

We prove many further related results and many of our results are proved by the Stability Method. We have several further results in this area. Altogether, mostly we restricted ourselves here to the simplest versions of our results, and left out many important related results, to be mentioned in my lecture. This lecture is partly based on a manuscript of Erdős and Simonovits [3] from the late 1980s.

References

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[4] Erdős, P., M. Simonovits and V. T. Sós: Anti-Ramsey theorems, Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II, pp. 633643, Colloq. Math. Soc. Jnos Bolyai, Vol. 10, North- Holland, Amsterdam, 1975.

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